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Asymptotic Behavior of the Global Solution for a Singularly Perturbed KdV Equation with Initial Value

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Abstract— **Making use of Fourier techniques, this paper deals with the global solution of a singularly perturbed KdV equation with initial value. Under certain assumption, The existence and uniqueness of the global solution to the singularly perturbed KdV equation is gained in Sobolev space.And the long time asymptotic behaviour of the form a approximation solution is discussed.**

Key words— **KdV equation; initial value problem; global solution; asymptotic solution.**

INTRODUCTION

Recently there has been much interestin the global solution of non-linear developing equation, it has been an important object in physics, mechanics, biology. Many applied mathematics workers and explorers working in Turing Machine spend their time in looking for the global solution of some non-linear partial differential equations.The generated KdV function

$$u_t + 6uu_x + u_{xxx} + u_{xxxxx} = 0 \quad \ldots\ldots\ldots\ldots(1)$$

is an important function which is often seen in physics, wave mechanics, biology, chemistry, and so on.

In this paper, we will explore the global solution of the initial value problem (1) and the global solution is asymptotic in long time.

MAIN RESULT

First ,we give the initial value problem:

$$\begin{cases} u_t + 6uu_x + u_{xxx} + u_{xxxxx} = 0, \\ u(0,x,\delta) = \delta\phi(x). \end{cases} \quad \ldots\ldots\ldots (2)$$

where $t \in \mathrm{R}$, $x \in \mathrm{R}$, $\varphi \in H^s(\mathrm{R}), s > \frac{1}{2}$.

In order to proof the theorem, we should give a definition.

Definition 1 Let

$$X_s(T) = C([0,T], H^{s+1}(R)) \bigcap C^1([0,T], H^s(R)).$$

The normal is

$$\| u \|_{X_s(T)} = \sup_{t\in(0,T)} (\| u \|_{s+1} + \| u_t \|_s)$$

Where

$$\| u \|_s = \left(\int_{-\infty}^{+\infty} (1 + |\lambda|^2)^s \, |\hat{u}(x,t)|^2 \, d\lambda \right)^{\frac{1}{2}}.$$

Let

$X_s(\infty) = C([0,\infty], H^{s+1}(\mathrm{R})) \bigcap C^1([0,\infty], H^s(\mathrm{R}))$. If $s > \frac{1}{2}$, according to the theorem of the Sobolev Space we know

$$\| uv \|_s \le C \| u \|_s \| v \|_s .$$

In the following we act C as all instant numbers.

Theorem 1 If $\varphi \in H^s(\mathrm{R}), s > \frac{1}{2}$, $0 < \delta < \delta_0 << 1$, there exits the unique global solution of the initial value problem, where $u(x,t) \in X_S(\infty)$.

Proof : Making the Fourier transform about x in the initial value problem (2), we will get

$$\begin{cases} \hat{u}_t + 3(\hat{u}^2)_x - i\lambda^3 \hat{u} + i\lambda^5 \hat{u} = 0, \\ \hat{u}(0,\lambda) = \delta \hat{\varphi}(\lambda). \end{cases} \quad \ldots\ldots\ldots (3)$$

And then

$$\hat{u} = \mathrm{e}^{(i\lambda^3 - i\lambda^5)t} \delta \hat{\phi}(\lambda) - i\lambda \mathrm{e}^{(i\lambda^3 - i\lambda^5)t} \int_0^t \hat{u^2}_x \mathrm{e}^{(i\lambda^3 - i\lambda^5)(t-\tau)} d_\tau .$$

Let $\hat{u}_0 = \mathrm{e}^{(i\lambda^3 - i\lambda^5)t} \delta \hat{\varphi}(\lambda)$.

We will get

$$\|u_0\|_s \le \delta \|\phi\|_s = \delta A, \qquad (4)$$

where $A = \|\varphi\|_s$.

We definite the sequence $\{\hat{u}_n\}$:

$$\hat{u}_n = \hat{u}_0 - e^{(i\lambda^3 - i\lambda^5)t} \int_0^t (\hat{u}_{n-1}^2)_x e^{(i\lambda^3 - i\lambda^5)(t-\tau)} d_\tau$$

where $n = 1,2,3,\cdots$

In the following we will proof that there is the function $u(x,t) \in X_s(\infty)$, and the sequence $\{u_n\}$ will be uniformly convergent to $u(x,t)$.

As we all know that there are three important conservation laws in physics. When a physics problem can be made by a non-linear partial differential equation $u_t = k(u)$, according to conservation laws we can get

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0.$$

where T and X is polynomial for $u(x,t)$. And then $I = \int T dx$ have no relations of time t.

For the singularly perturbed KdV equation,we can attain the result as follow

$$[u^2]_t + [4u^3 + 2uu_{xx} - u_x^2 + \varepsilon u_{xxxxx}]_x = 0.$$

$$[-\frac{1}{2}u_x^2 + u^3]_t + [-6uu_x^2 + 3u^2 u_{xx} - u_x u_{xxx} + \frac{1}{2}(u_{xx})^2 - \varepsilon u_x u_{xxxxx} + \varepsilon u_{xx} u_{xxxxx} - \frac{1}{2}\varepsilon(u_{xxx})^2 + \frac{9}{2}u^4 + u^2 u_x + \frac{1}{2}\varepsilon u^2 u_{xxxx} - \varepsilon u u_x u_{xxx} + \frac{1}{4}\varepsilon u_x^2 u_{xx} - \frac{7}{12}\varepsilon u_x^3 + \frac{1}{2}\varepsilon u u_{xx}^2]_x = 0.$$

Define $\Phi_0(u) = \int_R u^2(x,t)dx$,

$$\Phi_1(u) = \int_R (-\frac{1}{2}u_x^2 + u^3)dx.$$

It is easy to know

$$\frac{d}{dt}\Phi_j(u) = 0 \quad j = 0,1 \quad t \in R.$$

And for the random time $t \in R$,we know

$$\int_R u^2(x,t)dx = \int_R \varphi^2(x)dx$$

$$\int_R (-\frac{1}{2}u_x^2 + u^3)dx = -\frac{1}{2}\int_R \varphi_x^2 dx + \int_R \varphi^3 dx \quad \ldots..(5)$$

$$\|u_x\|^2 = \int_R u_x^2(x,t)dx = \int_R \phi_x^2 dx - 2\int_R \phi^3 dx + 2\int_R u^3 dx \qquad (6)$$

$$\|u_x\|^2 \le \|\phi_x\|^2 + 2\|\phi\|_1 \|\phi\|^2 + 2\|u\|_1 \|u\|^2 \qquad (7)$$

Since the theorem in Sobolev Space,we attain

$$\|u^3\|_{H^s(R)} \le C\|u\|^3_{H^s(R)}, s \ge \frac{1}{2}. \qquad (8)$$

For (4)(6)(7)and (8), we know

$$\|u_0{}^2{}_x\|_s = \|2u_0(u_0)_x\|_s \le 2\|u_0\|_s \|(u_0)_x\|_s \le 2\delta A \|(u_0)_x\|_s$$

$$\|(u_0)_x\|_s^2 \le \|\phi_x\|_s^2 + 2\|\phi\|_1 \|\phi\|_s^2 + 2\|u_0\|_1 \|u_0\|_s^2 \le C^2\delta^2 A^2$$

And then

$$\|u_1\|_s \le \delta A + \|u_0\|_s \|(u_0)_x\|_s \le \delta A + 2\delta A \cdot C\delta A. \quad .$$

If δ is small enough, we know

$$\|u_1\|_s \le 2\delta A.$$

According to induction,we know

$$\|u_n\|_s \le 2\delta A, n = 1,2,3,\cdots \qquad (9)$$

Using the way as (9),we attain

$$\|u_n\|_{X_s(T)} \le 2\delta A, \quad n = 1,2,3,\cdots$$

And then

$$\|u_n - u_{n-1}\|_{X_s(T)} \le \|u_{n-1}{}^3 - u_{n-2}{}^3\|_{X_s(T)} \le C[3(2\delta A)^2]\|u_{n-1} - u_{n-2}\|_{X_s(T)} \le [3C(2\delta A)^2]^{n-1}(\delta A).$$

$$\|u_n\|_{X_s(T)} \le \|u_0\|_{X_s(T)} + \sum_{i=1}^n \|u_i - u_{i-1}\|_{X_s(T)} \le \delta A + \sum_{i=1}^n [3C(2\delta A)^2]^{i-1}(\delta A)$$

So there exists a function

$$u(x,t) \in X_S(\infty).$$

And the sequence $\{u_n\}$ is uniformly convergent to $u(x,t)$. In the other way, $u(x,t) \in X_S(\infty)$ is the global solution of the initial value problem (2).

Next we proof the solution of the initial value problem (2) is unique.

Theorem 2 The solution of the initial value problem (2) is unique.

Proof: Suppose the initial value problem (2) has two solutions as $u^*(x,t) \in X_S(\infty)$,

and $u^{**}(x,t) \in X_S(\infty)$.

Let $\omega(x,t) = u^*(x,t) - u^{**}(x,t)$,

And then

$$\hat{\omega}(x,t) = -e^{(i\lambda^3 - i\varepsilon\lambda^5)t}\int_0^t (\hat{u}^{*2} - \hat{u}^{**2})_x e^{(i\lambda^3 - i\varepsilon\lambda^5)(t-\tau)} d_\tau$$

$$= -e^{(i\lambda^3 - i\varepsilon\lambda^5)t}\int_0^t [(\hat{u}^* + \hat{u}^{**})(\hat{u}^* - \hat{u}^{**})]_x e^{(i\lambda^3 - i\varepsilon\lambda^5)(t-\tau)} d_\tau$$

$$|\hat{\omega}(x,t)|$$

$$= |-e^{(i\lambda^3 - i\varepsilon\lambda^5)t}\int_0^t [(\hat{u}^* + \hat{u}^{**})(\hat{u}^* - \hat{u}^{**})]_x e^{(i\lambda^3 - i\varepsilon\lambda^5)(t-\tau)} d_\tau$$

$$\le |-e^{(i\lambda^3 - i\varepsilon\lambda^5)t}| [\int_0^t |(\hat{u}^* + \hat{u}^{**})_x (\hat{u}^* - \hat{u}^{**})| d\tau$$

$$+ \int_0^t |(\hat{u}^* + \hat{u}^{**})(\hat{u}^* - \hat{u}^{**})_x| d\tau]| \int_0^t e^{(i\lambda^3 - i\varepsilon\lambda^5)(t-\tau)} d_\tau|$$

$$\le C2[\int_0^t |\hat{\omega}(x,t)| d\tau + \int_0^t |\hat{\omega}(x,t)|_x d\tau]$$

$$\le C^* \int_0^t |\hat{\omega}(x,t)| d\tau$$

We can attain the similar result

$$\|\hat{\omega}(x,t)\|_{x_S(\infty)} \le C^* \int_0^t \|\hat{\omega}(x,t)\|_{x_S(\infty)} d\tau.$$

Since Gronwall inequation, we know

$$\omega(x,t) = 0(\omega(x,t) \in X_s(\infty)).$$

So in the $X_s(\infty)$ space, we get

$$u^*(x,t) = u^{**}(x,t).$$

Theorem 3 If the condition of the theorem 1 is existing, the unique solution of the initial value problem (2) can be written as

$$u(x,t) = \sum_{n=0}^{\infty} \delta^{n+1} u^{(n)}(x,t), \quad \ldots\ldots\ldots..(10)$$

where $\delta > 0$ and it is small enough.

Proof: Suppose the global solution of the initial value problem (2) as

$$u(x,t) = \sum_{n=0}^{\infty} \delta^{n+1} u^{(n)}(x,t) \qquad (11)$$

According to the initial value problem (2)

$$\begin{cases} \sum_{n=0}^{\infty} \delta^{n+1} \hat{u}_t^{(n)} + i(\varepsilon\lambda^5 - \lambda^3)\sum_{n=0}^{\infty} \delta^{n+1} \hat{u}^{(n)} \\ = -3[\sum_{n=0}^{\infty} \delta^{n+1} \hat{u}^{(n)}]^2_x \\ \sum_{n=0}^{\infty} \delta^{n+1} \hat{u}^{(n)}(0,\lambda) = \delta\hat{\phi}(\lambda) \end{cases}$$

Since the same power of δ is equal, when $n = 0$, we attain

$$\begin{cases} \hat{u}_t^{(0)}(x,\lambda) + i(\varepsilon\lambda^5 - \lambda^3)\hat{u}^{(0)} = 0, \\ \hat{u}^{(0)}(0,\lambda) = \hat{\phi}(\lambda). \end{cases} \quad \ldots\ldots\ldots..(12)$$

And then

$$\begin{cases} \hat{u}_t^{(0)}(x,\lambda) + i(\varepsilon\lambda^5 - \lambda^3)\hat{u}^{(0)} = 0, \\ \hat{u}^{(0)}(0,\lambda) = \hat{\phi}(\lambda). \end{cases}$$

So $\hat{u}^{(0)}(x,\lambda) = \hat{\phi}(\lambda)e^{i(\lambda^3 - \varepsilon\lambda^5)t}$.

We can know $\|\hat{u}^{(0)}\|_{x_s(\infty)} \le C\|\hat{\varphi}\|_{x_s(\infty)}$.

According to (5), when m is even number, we know

$$\begin{cases} \hat{u}_t^{(m)}(x,\lambda) + i(\varepsilon\lambda^5 - \lambda^3)\hat{u}^{(m)} \\ = -3[\hat{u}_x^{(0)}\hat{u}_x^{(m)} + \hat{u}_x^{(1)}\hat{u}_x^{(m-1)} + \cdots + \hat{u}_x^{(\frac{m}{2})}\hat{u}_x^{(\frac{m}{2})}] \\ \hat{u}^{(0)}(0,\lambda) = \hat{\phi}(\lambda) \end{cases}$$

The computation as (12), we know

$$\| \hat{u}^{(m)}(x,\lambda) \|_{x_s(\infty)} \le Cm \| \hat{\varphi} \|_{x_s(\infty)} .$$

When m is odd number, we can attain the similar result

$$\| \hat{u}^{(m)}(x,\lambda) \|_{x_s(\infty)} \le Cm \| \hat{\varphi} \|_{x_s(\infty)} \quad \ldots\ldots\ldots(13)$$

$$\|u(\hat{x},t)\|_{x^s(\infty)} \le \sum_{n=0}^{\infty} \delta^{n+1} \|u^{(n)}(\hat{x},t)\|_{x^s(\infty)}$$

$$\le \sum_{n=0}^{\infty} \delta^{n+1} Cn \|\hat{\phi}\|_{x_s(\infty)} \le C \|\hat{\phi}\|_{x_s(\infty)} B. \quad \ldots\ldots.(14)$$

Since $\delta \le \frac{1}{2}$, $\sum_{n=0}^{\infty} \delta^{n+1} n$ is convergent.

Let $\sum_{n=0}^{\infty} \delta^{n+1} n \le B$, where B is a constant. Since (14), (10) is unformly convergent. Using the reverse Fourier transform, the global solution of the initial value problem (2) can be expressed by (10).

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REFERENCES

[1] Huabing Jia,Wei Xu. Solitons solutions for some nonlinear evolution equations. Applied Mathema-
tics and Computation. 217(2010)1678-1687

[2] Abdul-Majid Wazwaz. New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations. Applied Mathematics and Computation. 182(2006) 1624-1650.

[3] Kaya,Salah M. El-Sayed. On a generalized fifth order KdV equations.Physics Letters A . 310 (2003) 44–51.

[4] Wenhua Hai, Yi Xiao. Soliton solution of a singularly perturbed KdV equation. Physics Letters A. 208 (1995) 79-83.

[5] Abdul-Majid Wazwaz. The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. Applied Mathematics and Computation. 184 (2007) 1002–1014.