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Computable Analysis of the Solution of the Nonlinear Kawahara Equation

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***Abstract*— In this paper, we study the computability of the solution operator of the initial problem for the nonlinear Kawahara equation, which is based on the Type-2 Turing machines. We will prove that in Sobolev space $H^s(R^2)$, for $s \geq 0$, the solution operator:**

$$K_R : H^s(R^2) \to C(R; H^s(R^2))$$

is $(\delta_{H^s}, [\rho \quad \to \delta_{H^s}])$ -computable. The conclusion enriches the theory of computability.

***Key words*— the nonlinear Kawahara equation, initial problem, computability, Type-2 theory of effectivity (TTE), Sobolev space**

Ⅰ.INTRODUCTION

At present, the computability of solutions of the nonlinear evolution equations have become an important topic to the workers of physics, mechanics, life science, applied mathematics, engineering and theoretical computer. Researching boundedness and computability of the solutions of the nonlinear equations, will offer effective tools for the application of equations, enrich theoretical foundation of computer science and promote the development of computer software. In 1985, K.Weihrauch and others established a computational model, called Type-2 theory of effectivity (TTE for short). K.Weihrauch and N. Zhong have studied the computability of the generalized functions, the KdV equation and the Schrödinger equation [3]–[5], Dianchen Lu and others have studied the computability of the mKdV equation [1].

The nonlinear Kawahara equation was first proposed by Kawahara in 1972, this equation has wide applications in physics such as in the theory of magneto–acoustic waves in plasma, in the theory of long waves in shallow liquid under ice cover and so on. In this paper, we will discuss the nonlinear Kawahara equation as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^5 u}{\partial x^5}\right) = -\frac{1}{2}c_0\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$u(x,y,0) = \phi(x,y), x, y \in R, t \geq 0 \tag{2}$$

where $\beta, \gamma \in R$ are dispersion coefficients, a is nonlinear perturbance coefficient, and $c_0 > 0$ is sound velocity.

The paper is organized as follows. In Section2, we mainly review some basic definitions, lemmas and conclusions of TTE, which are relevant to the proof of section3. Section3 is devoted to the proof of the main theorem.

Ⅱ. PRELIMINARIES

This section we will give a brief introduction of TTE. For details the reader can refer to [2].

Lemma 2.1

1) In Schwarz space $S(R)$, the function $(a, \psi) \mapsto a\psi$ is $(\rho, \delta_s, \delta_s)-$ computable; $(\psi, t) \mapsto |\psi(t)|$ is (δ_s, ρ, ρ) $-$ computable; $(\phi, \psi) \mapsto \phi + \psi$ and $(\phi, \psi) \mapsto \phi \cdot \psi$ are $(\delta_s, \delta_s, \delta_s)-$ computable.

2) The function $(\psi,t)\mapsto E_m(t)\cdot\psi,\ (\psi,t)\mapsto E_m(t)\cdot\psi,$ is $(\delta_s,\rho,\delta_s)-$computable, for computable $m\in R$.

3) The Fourier transform

$F:S(R)\to S(R),\ \varphi\mapsto(2\pi)^{-1/2}\int_R e^{-i\xi x}\varphi(x)dx$,

and the inverse Fourier transform

$F^{-1}:S(R)\to S(R),\varphi\mapsto(2\pi)^{-1/2}\int_R e^{i\xi x}\varphi(\xi)d\xi$,

are both $(\delta_s,\delta_s)-$computable.

Lemma 2.2

The function $H:C(R;S(R))\times R\times R\to S(R),$

$$H(u,a,b)=\int_a^b u(t)dt$$

is $([\rho\to\delta_s],\ \rho,\rho,\delta_s)-$computable.

Lemma 2.3 (type conversion)

Let $\delta_i:\subseteq\Sigma^\omega\to X_i$ be a representation of the set X_i $(0\le i\le k)$. Let $f:\subseteq X_1\times\cdots\times X_k\to X_0$ and define

$$L(x_1,\cdots,x_{k-1})(x_k):=f(x_1,\cdots,x_k),$$

then f is $(\delta_1,\cdots,\delta_k,\delta_0)-$computable (continuous) if and only if L is $(\delta_1,\cdots,\delta_{k-1},\ [\delta_k\to\delta_0])-$ computable (continuous).

Lemma 2.4 (primitive recursion)

Let $\gamma:\subseteq Y\to M$ and $\gamma':\subseteq Y\to M'$ are two representations, υ_N is admissible representation of N. Then we have the following propositions:

1) Suppose $f:\subseteq M\to M'$ is $(\gamma,\gamma')-$computable, $f':\subseteq N\times M'\times M\to M'$ is $(\upsilon_N,\gamma',\gamma,\gamma')-$ computable. Define $g':\subseteq N\times M\to M'$ as follows:

$$g'(0,x)=f(x),g'(n+1,x)=f'(n,g'(n,x),x),$$

where $x\in M,n\in N$, then the function g' is $(\upsilon_N,\ \gamma,\gamma')-$computable.

2) Suppose $h:\subseteq M\to M$ is $(\gamma,\gamma)-$computable, define a function $H:\subseteq N\times M\to M$ as follows:

$$H(0,x)=x,\ H(n+1,x)=h\circ H(n,x)=h^{n+1}(x),$$

then the function H is $(\upsilon_N,\gamma,\gamma)-$computable.

The conclusions about the computability above also can apply to multidimensional space $R^n(n\ge 2)$.

Definition2.5[6] For a fixed $T>0$ and any function $\omega(x,y,t):[0,T]\times R^2\mapsto R$, define modular functions:

$$\Lambda_1^T(\omega)=\sup_{[0,T]}\|\omega\|_{s,2} \tag{3}$$

$$\Lambda_2^T(\omega)=\left\|\frac{\partial\omega}{\partial x}\right\|_{L_t^2L_{(x,y)}^\infty} \tag{4}$$

$$\Lambda^T(\omega)=\max\{\Lambda_1^T(\omega),\Lambda_2^T(\omega)\} \tag{5}$$

Where $\|\cdot\|_{s,2}$ means the norm of Sobolev space $W^{s,2}(R^2)$, which is also the norm of $H^s(R^2)$, we construct a function space as follows:

$X_T=\{\omega\in C([0,T];H^s(R^2)),s\ge 0,\Lambda^T(\omega)<\infty\}$,

Then for any $u\in X_T$, it holds that

$$\left\|u\frac{\partial u}{\partial x}\right\|_{L_t^2L_{(x,y)}^2}\le(\Lambda^T(u))^2 \tag{6}$$

Define the operator in $H_x^{-1}(R^2)$, $\partial_x^{-1}: F(\partial_x^{-1} f) = (1/(ik_1))\hat{f}$.

Where $H_x^{-1}(R^2) = \{ f \in \phi'; \frac{1}{k_1}\hat{f}(k_1,k_2) \in L^2(R^2)\}$,

$$\|f\|_{H_x^{-1}} = \left\|\frac{1}{k_1}\hat{f}\right\|_{L^2(R^2)}.$$

Then the Cauchy problem (1)-(2) are equivalent to

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^5 u}{\partial x^5} + \frac{1}{2}c_0\partial_x^{-1}(\frac{\partial^2 u}{\partial y^2}) = 0 \quad (7)$$

$$u(x,y,0) = \phi(x,y), x,y \in R, t \ge 0 \quad (8)$$

The linear part of (7)-(8) is

$$\frac{\partial u}{\partial t} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^5 u}{\partial x^5} + \frac{1}{2}c_0\partial_x^{-1}(\frac{\partial^2 u}{\partial y^2}) = 0 \quad (9)$$

$$u(x,y,0) = \phi(x,y), x,y \in R, t \ge 0 \quad (10)$$

Then use the Fourier transform, we obtain the solution of (9)-(10) is

$$G(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x\xi_1 + y\xi_2)} e^{it(\gamma\xi_1^5 + \beta\xi_1^3 - (1/2)c_0\frac{\xi_2^2}{\xi_1})} d\xi_1 d\xi_2 \quad (11)$$

Note $$G_\alpha(x,y,t) = (-\frac{\partial^2}{\partial x^2})^{\frac{\alpha}{2}} G(x,y,t), (x,y) \in R^2 \quad (12)$$

Suppose $$W(t)\varphi(x) = (G(\cdot,t) * \varphi)(x) \equiv u(x,y,t) \quad (13)$$

where $u(x,y,t)$ is the solution of (9)-(10).

Let $W_{\alpha+i\theta}(t)\phi(x,y) =$

$$\int_{R^2} |\xi_1|^{\alpha+i\theta} e^{it(\gamma\xi_1^5 + \beta\xi_1^3 - (c_0/2)\xi_2^2/\xi_1)} \cdot e^{i(x\xi_1 + y\xi_2)}\hat{\phi}(\xi_1,\xi_2) d\xi_1 d\xi_2 \quad (14)$$

When $\theta = 0$, we have

$$W_\alpha(t)\phi(x,y) = (-\frac{\partial^2}{\partial x^2})^{\frac{\alpha}{2}} W(t)\phi(x,y)$$

$$= (-\frac{\partial^2}{\partial x^2})^{\frac{\alpha}{2}} u(x,y,t) = (G_\alpha(\cdot,t) * \phi)(x)$$

Next we will give some lemmas and theorems about estimators.

***Theorem 2.6*[6]** The initial value problem of (9)-(10) is given by (11), then for any fixed $T > 0$, when $0 \le \alpha \le 1$, for any parameters $\gamma \ne 0, \beta, c_0 > 0$,there exist an constant $C > 0$, which only depends on α and parameters γ, β, c_0, then it holds that

$$\left|(-\frac{\partial^2}{\partial x^2})^{\frac{\alpha}{2}} G(x,y,t)\right| \le CT^{\frac{\alpha}{5}} t^{-\frac{\alpha+4}{5}}, 0 \le t \le T \quad (15)$$

***Lemma 2.7*[6]** For any $0 \le \alpha \le 1$, the estimator

$$\|W_\alpha(t)\varphi\|_{L^{2(1-\alpha)}(R^2)} \le CT^{\frac{\alpha}{5}} t^{-\alpha} \|\varphi\|_{L^{2/(1+\alpha)}(R^2)} \quad (16)$$

for any fixed $T > 0, 0 < t \le T$ holds，where C is positive constant, which only depends on $\gamma, \beta, c_0, \alpha$.

***Theorem 2.8*[6]** Suppose $0 \le \alpha \le 1$, $p = \frac{2}{1-\alpha}$, $q = \frac{2}{\alpha}$. $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$, then for any fixed $T > 0$, there exists C which depends on $\gamma, \beta, c_0, \alpha$, it holds that

$$\left\|W_{\frac{\alpha}{2}}(t)\phi\right\|_{L_t^q L_{(x,y)}^p} \le CT^{\frac{\alpha}{5}} \|\phi\|_{L^2(R^2)} \quad (17)$$

And for any $g \in L_t^{q'} L_{(x,y)}^{p'}$, we have

$$\left\|\int_0^t W_\alpha(t-\tau) g(\cdot,\cdot,\tau) d\tau\right\|_{L_t^q L_{(x,y)}^p} \le CT^{\frac{\alpha}{5}} \|g\|_{L_t^{q'} L_{(x,y)}^{p'}} \quad (18)$$

III. MAIN RESULT

Theorem 3.1 The initial value problem (7)-(8) define a nonlinear map $K_R : H^s(R^2) \to C(R; H^s(R^2))$, which is from the initial value φ to the solution u, and the solution operator is $(\delta_{H^s}, [\rho \to \delta_{H^s}]) -$ computable.

If the initial value problem of (7) is $u(x,y,t_0) \in H^s(R^2)$, then in the neighborhood of t_0, apply contracting mapping principle , the initial value problem of (7) can be constructed, usually we begin this from $t_0 = 0$. Suppose the solution $u(x,y,t)$ has been constructed in the interval $[0,t_0]$. Next we will prove how to extend the solution from the neighborhood $t = t_0$. Next we consider the equation with the initial value $v(t_0)$:

$$\frac{\partial v}{\partial t} + av\frac{\partial v}{\partial x} + \beta\frac{\partial^3 v}{\partial x^3} + \gamma\frac{\partial^5 v}{\partial x^5} + \frac{1}{2}c_0\partial_x^{-1}(\frac{\partial^2 v}{\partial y^2}) = 0 \quad (19)$$

$$v(x,y,t_0) = \psi(x,y), \psi \in H^s(R^2) \quad (20)$$

The equivalence integral equation of (19)-(20) is

$$v(t) = W(t-t_0)\psi + a\int_{t_0}^{t} W(t-\tau)(v\frac{\partial v}{\partial x})(\tau)d\tau \quad (21)$$

Define two maps A and G ,

$\forall t_0 \in R^+, t - t_0 \le T, G(t_0) : C(R; H^s(R^2)) \to C(R;$ $H^s(R^2))$ is defined

$$G(t_0)(\eta)(t) = a\int_{t_0}^{t} W(t-\tau)(\eta\frac{\partial \eta}{\partial x})(\tau)d\tau \quad (22)$$

$\forall t_0 \in R^+, t - t_0 \le T, \psi \in H^s(R^2), A(t_0) : C(R;$ $H^s(R^2)) \to C(R; H^s(R^2))$ is defined

$$A(t_0,\psi)(\eta)(t) = W(t-t_0)\psi + G(t_0)(\eta)(t) \quad (23)$$

Let $X_T^b = \{u \in X_T; \Lambda^T < b\}$ is a ball of X_T and the radius is b, then there exist $T = T(\|\psi\|_{s,2}, \gamma, \beta, c_0, a) > 0$ and $b = b(\|\psi\|_{s,2}, \gamma, \beta, c_0, a) > 0$, which make the maps A and G are contracting in the neighborhood of t_0.

Form (17), we have

$$\Lambda^T(W(t-t_0)\psi) \le C\|\psi\|_{s,2} \quad (24)$$

When $\alpha = 0$ in (18) and combine with (6), we have

$$\Lambda_1^T(G(t_0)(\eta)(t))$$

$$\le aCT^{\frac{1}{2}}(\int_{t_0}^{t_0+T}\int_{R^2}\left|\eta(x,y,t)\frac{\partial\eta}{\partial x}(x,y,t)\right|^2 dxdydt)^{\frac{1}{2}}$$

$$\le aCT^{\frac{1}{2}}(\Lambda^T(\eta))^2 \quad (25)$$

Where C is a constant depends on γ, β, c_0. Likewise, when $\alpha = 1$ in (18), we have

$$\Lambda_2^T(G(t_0)(\eta)(t)) = a\left\|\int_{t_0}^{t} W_1(t-\tau)(\eta\frac{\partial\eta}{\partial x})(\tau)d\tau\right\|_{L_t^2 L_{(x,y)}^\infty}$$

$$\le aCT^{\frac{1}{5}}\left\|\eta\frac{\partial\eta}{\partial x}\right\|_{L_t^2 L_{(x,y)}^1}$$

$$\le aCT^{\frac{1}{5}}\int_{t_0}^{t_0+T}(\int_{R^2}\left|\eta(x,y,t)\frac{\partial\eta}{\partial x}(x,y,t)\right|^2 dxdy)^{\frac{1}{2}}dt$$

$$\le aCT^{\frac{1}{5}}\cdot T^{\frac{1}{2}}(\int_{t_0}^{t_0+T}\int_{R^2}\left|\eta(x,y,t)\frac{\partial\eta}{\partial x}(x,y,t)\right|^2 dxdydt)^{\frac{1}{2}}$$

So from (6), we have

$$\Lambda_2^T(G(t_0)(\eta)(t)) \le aCT^{\frac{7}{10}}(\Lambda(\eta))^2 \quad (26)$$

From (24), (25), (26) and combine with (21), it holds that

$$\Lambda^T(A(t_0,\psi)(\eta)(t))$$

$$\le C\|\psi\|_{s,2} + aC\max\{T^{\frac{1}{2}}, T^{\frac{7}{10}}\}(\Lambda^T(\eta))^2 \qquad (27)$$

Note $\theta = \max\{T^{\frac{1}{2}}, T^{\frac{7}{10}}\}$, then for any fixed initial value $\psi(x,y)\in H^s(R^2)(s\ge 0)$, let $b = 2C\|\psi\|_{s,2}$ and seek T, it holds that

$$4Cab\theta < 1 \qquad (28)$$

From(27),we have $\Lambda^T(A(t_0,\psi)(\eta)(t)) \le \frac{1}{2}b + \frac{1}{4}b < b$ (i.e. $\Lambda^T(A(t_0,\psi))\in X_T^b$), then for any $\eta_1,\eta_2 \in X_T^b$,

$$\Lambda^T(A(t_0,\psi)(\eta_1) - A(t_0,\psi)(\eta_2))$$

$$= \Lambda^T(G(t_0)(\eta_1) - G(t_0)(\eta_2))$$

$$\le aC\theta(\Lambda^T(\eta_1-\eta_2))^2$$

$$\le \frac{1}{2}\Lambda^T(\eta_1-\eta_2) \qquad (29)$$

On the basis of (21) and (29), the fixed point in the contracting map $A(t_0,\psi)$ satisfies $v(x,y,t_0)=\psi$, and is the solution of the initial value of the integral equation (7). Therefore, if we can compute A in $H^s(R^2)$ space, then we can compute the solution of initial value in the following lemma will show that the restriction of A on the Schwarz space $S(R^2)$, a dense subset of $H^s(R^2)$, is computable. This restriction will also be denoted as A.

Lemma 3.2 The restriction of the operator A to $S(R^2)$: $R\times S(R^2)\times C(R;S(R^2)) \to C(R;S(R^2))$, $(t_0,\psi,\eta) \to A(t_0,\psi)(\eta)(t)$ is $(\rho,\delta_s,[\rho\to\delta_s],[\rho\to\delta_s])-$ computable.

Proof. By lemma 2.1 and 2.2, the function:

$$(t_0,\psi,\eta) \to A(t_0,\psi)(\eta)(t)$$

is $(\rho,\delta_s,[\rho\to\delta_s],\ \rho,\delta_s)-$ computable.

Lemma 3.3 The map $F_1:(t_0,\psi,n)\to(A(t_0,\psi))^n(0)$ is $(\rho,\delta_s,v_N,[\rho\to\delta_s])-$ computable.

Where $(A(t_0,\psi))^n(\eta)) = A(t_0,\psi)((A(t_0,\psi))^{n-1}(\eta))$ is the nth iteration.

Proof. By Lemma 2.4 and A is computable, we can get it.

Next we prove $A(t_0,\psi)$ is computable in $H^s(R^2)$. Let u_ψ is the fixed point of $A(t_0,\psi)$, construct a sequence of iterations as follows:

$$u_{\psi_k}^{n+1} = A(0,\psi_k)^{n+1}(\eta) = A(0,\psi_k)(A(t_0,\psi_k)^n(\eta)).$$

Select a subsequence of natural number $\{j_i\}$, which satisfies $\Lambda^T(u_{\psi_k}^{j^i} - u_{\psi_k}) \le 2^{-k-1}$. Because $S(R^2)$ is dense in $H^s(R^2)$, so there exists $\psi_k\in S(R^2)$ satisfies $\|\psi_k-\psi\|_{s.2}\le 2^{-k-2}$. Let $\frac{b}{2\|\psi\|_{s,2}}\le 2$, so we have

$$\Lambda^T(u_\psi - A(t_0,\psi_k)(\eta)) = \Lambda^T(A(t_0,\psi)(\eta) - A(t_0,\psi)(\eta))$$

$$= \Lambda^T(W(t-t_0)\psi - W(t-t_0)\psi_k)$$

$$= \Lambda^T(W(t-t_0)(\psi-\psi_k))$$

$$\le C\|\psi-\psi_k\|_{s,2} = \frac{b}{2\|\psi\|_{s,2}}\|\psi-\psi_k\| \le 2^{-k-1}$$

$$\Lambda^T(u_\psi - A(t_0,\psi_k)^{j_i}(\eta))$$

$\leq \Lambda^T(u_\psi - A(t_0,\psi_k)(\eta)) + \Lambda^T(A(t_0,\psi_k)(\eta) - A(t_0,\psi_k)^{j_i}(\eta))$

$\leq 2^{-k-1} + \Lambda^T(u_{\psi_k}^{j_i} - u_{\psi_k}) \leq 2^{-k}$

Lemma 3.4 The map

$$F_+ : (t_0, \psi, t) \to v(t), t \in [t_0, t_0 + \theta]$$

is $(\rho, \tilde{\delta}_{H^s}, \rho, \tilde{\delta}_{H^s}) -$ computable.

Because $\psi = u(t_0), v(t)$ is the solution of (7) in the interval $[t_0, t_0 + \theta]$, so the solution is extended form $[0, t_0]$ to $[t_0, t_0 + \theta]$.

Suppose a, β, γ, c_0, T are computable real numbers, and $\|\psi\|_{s,2}$ is computable, so C, θ, b are computable.

For integer $z \in Z$, we compute the solution $u(z \cdot \theta)$ at times $z \cdot \theta$. Define

$$H_+(\varphi, 0) = H_-(\varphi, 0) = \varphi$$
$$H_+(\varphi, n+1) = F_+(n \cdot \theta, H_+(\varphi, n), (n+1) \cdot \theta)$$

By Lemma 3.4 $H_+(\varphi, n) = u(n \cdot \theta)$ is computable from n and φ, because H_+ is primitive recursion of the function F_+.

From t, θ, we can compute $z \in Z$, such that $z\theta \leq t \leq (z+1)\theta$. Apply Lemma 3.4, we can compute $u(z \cdot \theta)$ and $F_+(z \cdot \theta, u(z \cdot \theta), t)$, where $F_+(z \cdot \theta, u(z \cdot \theta), t) = u(t)$.

In Lemma 3.4, we can prove that for $t_0 = 0$, the solution of the initial value problem of (7)-(8) is computable,.

So we prove the Theorem 3.1.

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