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The M/M/c Queue with Setup Times and Single Working Vacations of Partial Servers

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***Abstract*— In this paper, we investigate an M/M/c queue with single working vacations of partial servers and setup times. During the vacation period, some of which servers are not entirely stop the service but service the new customers at a lower service rate, the other servers have a normal vacation and stop the service. Quasi-birth-and-death-process and infinitesimal generators for the process are obtained from the model that has been described. The steady-state distributions of queue length and some system characteristics are obtained by using matrix-geometric solution method.**

***Keywords*— Multi-server queue, set-up time, partial working vacations, quasi birth and death process, matrix-geometric solution, conditional stochastic decompositions.**

I. Introduction

The earlier studies of multi-server vacation queue system can be traced back to Levy, Yechiali[1] and Vinod[2]. They studied the M/M/c queue with exponential vacation time, but failed to give the distribution of steady state indicators. In recent years, the authors of the literature [3-7] have been studied a variety of vacation policy in M/M/c and GI/M/c queue system, not only obtained the distribution of queue length and waiting time at steady state, but also established the theoretical framework of conditional stochastic decomposition. [8] and [9] studied the M/M/c queue system with multiple vacation of partial servers and single vacation of partial servers. Further considering with working vacation policy based on the literature [9], we studied the M/M/c queue with setup times and single working vacations of partial servers. The steady-state distributions of queue length and some system characteristics are obtained by using matrix-geometric solution method.

II. System Description

A. Model Description

In the classic M/M/c queue with the arrival rate λ and the service rate μ_b , we introduced the synchronous single working vacation policy to partial servers. When there are no customers in the system, all servers start a vacation of random length synchronously, vacation time V have an exponential distribution with rate θ, during the vacation period, some of which $d(d<c)$ servers are not entirely stop the service but service the new customers at a lower service rate $\mu_v(\mu_v < \mu_b)$, the other c-d servers have a normal vacation and stop the service. At the end of the vacation, if there are customers in the system waiting for service, all the servers will immediately terminate the vacation and service the customers at the normal service rate until the system becomes empty, otherwise, the system begin a shut down period. If there are customers arriving at the period, the shut down period will be over. But the customers can not be serviced immediately, the system needs to go through a set-up period, the setup time S have an exponential distribution with rate α. After setup period, the regular busy period began. Our model is actually a conversion process among the three forms of high-speed service, low-speed service and complete stop service.

In order to save operating costs and improve efficiency, we set up a working vacation period and a shut down period. Essential difference between these two periods: customers in the former period can be served at a lower service rate; customers in the latter period cannot be served. Therefore, our model has an important practical significance to the optimization of design and control of the system.

Assume that the arrival interval time, service time, working vacation time and start-up time are all independent of each other. In addition, the service order is first in the first out (FIFO).

Let $Q(t)$ be the number of customers in the system at time t and

$$J(t)=\begin{cases}0, & \text{system in a working vacation at time } t\\ 1, & \text{system in a setup period and shut down period at time } t\\ 2, & \text{system in a regular busy period at time } t\end{cases}$$

So, $\{Q(t),J(t)\}$ is a Markov Process with a state space:

$$\Omega=\{(0,0),(0,1)\}\cup\{(k,j),k\ge 1, j=0,1,2\}$$

Among them, the state $(k, 0)$ $(k\geq0)$ indicate that the system is at working vacation period and there are k customers in the system, the state $(k, 1)$ $(k\geq1)$ indicate that the system is at setup time and there are k customers in the system, the state $(0,1)$ means that the system is shut down. The state $(k, 2)$ $(k\geq1)$ indicate that the system is at regular busy period and k customers in the system.

B. State Transition Diagram

According to the former description, the model state transition diagram is shown in fig. 1.

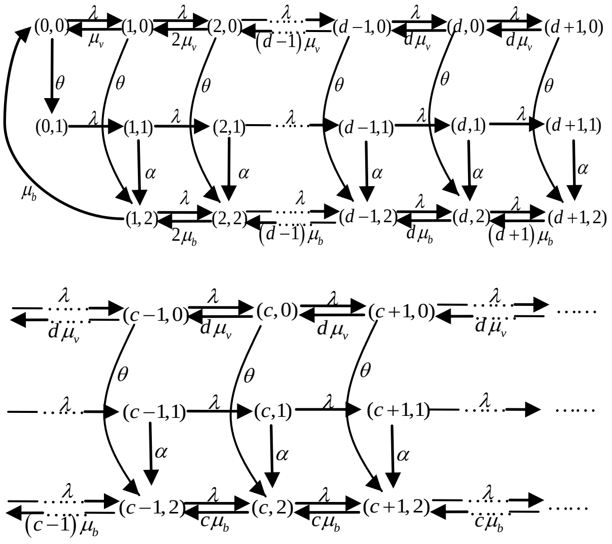

Fig. 1 State Transition Diagram

According to the state transition, we can come up with a Q-matrix based on the information of fig. 1, it is

$$Q=\begin{pmatrix} A_0 & C_0 & & & & & \\ B_1 & A_1 & C & & & & \\ & B_2 & A_2 & C & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & B_{c-1} & A_{c-1} & C & \\ & & & & B & A & C \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Where

$$A_0=\begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -\lambda \end{pmatrix}\quad C_0=\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}\quad B_1=\begin{pmatrix} \mu_\nu & 0 \\ 0 & 0 \\ \mu_b & 0 \end{pmatrix}$$

$$A_k=\begin{pmatrix} -(\lambda+\theta+k\mu_\nu) & 0 & \theta \\ 0 & -(\lambda+\alpha) & \alpha \\ 0 & 0 & -(\lambda+k\mu_b) \end{pmatrix}\quad 1\le k\le d,$$

$$A_k=\begin{pmatrix} -(\lambda+\theta+d\mu_\nu) & 0 & \theta \\ 0 & -(\lambda+\alpha) & \alpha \\ 0 & 0 & -(\lambda+k\mu_b) \end{pmatrix}\quad d+1\le k\le c.$$

$$B_k=\begin{pmatrix} k\mu_\nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k\mu_b \end{pmatrix}\ 2\le k\le d,\quad B_k=\begin{pmatrix} d\mu_\nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k\mu_b \end{pmatrix}\ d+1\le k\le c.$$

$$A=\begin{pmatrix} -(\lambda+\theta+d\mu_\nu) & 0 & \theta \\ 0 & -(\lambda+\alpha) & \alpha \\ 0 & 0 & -(\lambda+c\mu_b) \end{pmatrix}$$

$$B=\begin{pmatrix} d\mu_\nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c\mu_b \end{pmatrix}\qquad C=\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

III. Rate matrix and equilibrium conditions

In the analysis of QBD model, the solution of the quadratic equation of matrix

$$R^2B+RA+C=0 \tag{1}$$

called rate matrix plays an important role, to see Neuts(1981)[12], Tian Naishuo, YUE De-quan[13], or Latouche and Ramaswamy(1999)[14], in order to obtain the rate matrix, requires the following:

Lemma 1 The quadratic equation

$$d\mu_v z^2-(\lambda+\theta+d\mu_v)z+\lambda=0 \tag{2}$$

have different real roots r and r^*, and $0<r<1, r^*>1$.

Proof Discriminant of the equation

$$\Delta=(\lambda-d\mu_v)^2+\theta^2+2\theta(\lambda+d\mu_v)>0 \tag{3}$$

the quadratic equation has two real roots

$$r^*, r=\frac{\lambda+\theta+d\mu_v\pm\sqrt{(\lambda+\theta+d\mu_v)^2-4\lambda d\mu_v}}{2d\mu_v}$$

From (3), we easy to verify

$$(\lambda-d\mu_v+\theta)^2<\Delta<(\lambda+d\mu_v+\theta)^2, \lambda\ge d\mu_v$$

$$(d\mu_v-\lambda+\theta)^2<\Delta<(\lambda+d\mu_v+\theta)^2, \lambda<d\mu_v$$

substituting r and r^*, we obtain $0<r<1$ and $r^*>1$.

Theorem 1 When $\rho=\lambda(c\mu_b)^{-1}<1$, the matrix equation (1) have the minimal nonnegative solution

$$R=\begin{pmatrix} r & 0 & \dfrac{r\theta}{(1-r)c\mu_b} \\ 0 & \dfrac{\lambda}{\lambda+\alpha} & \rho \\ 0 & 0 & \rho \end{pmatrix} \tag{4}$$

Proof Because matrix $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are upper triangular matrix, we assume that

$$R=\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

substituting $\boldsymbol{R}$ into (1) we obtain the equations:

$$\begin{cases} d\mu_v r_{11}^2-(\lambda+\theta+d\mu_v)r_{11}+\lambda=0 \\ -(\lambda+\alpha)r_{12}=0 \\ c\mu_b(r_{11}r_{13}+r_{12}r_{23}+r_{13}r_{33})+\theta r_{11}+\alpha r_{12}-(\lambda+c\mu_b)r_{13}=0 \\ -(\lambda+\alpha)r_{22}+\lambda=0 \\ c\mu_b(r_{22}r_{23}+r_{23}r_{33})+\alpha r_{22}-(\lambda+c\mu_b)r_{23}=0 \\ c\mu_b r_{33}^2-(\lambda+c\mu_b)r_{33}+\lambda=0 \end{cases} \tag{5}$$

we mark (5.1~5.6) to each equation in (5), in order to get the minimal nonnegative solution, reference to Lemma 1, $r_{11}=r$ in equation (5.1), In equation (5.6), take $r_{33}=\rho$ (another root $r_{33}=1$), from (5.2) and (5.4), we have $r_{12}=0$, $r_{22}=\lambda(\lambda+\alpha)^{-1}$.

Substituting the above results into (5.3) and (5.4), we obtained $r_{13}=\theta r[c\mu_b(1-r)]^{-1}, r_{23}=\rho$.

Lemma 2 The rate matrix $\boldsymbol{R}$ satisfies $RT^0=\lambda e$, the column vector $T^0=(d\mu_v,0,c\mu_b)^\tau$, $e=(1,\cdots,1)^\tau$ "τ" indicates the matrix transpose.

Proof The column vector *e* right-multiply the both ends of the matrix equation (1), we have

$$R^2T^0-R(\lambda e+T^0)+\lambda e=0$$

that is

$$(I-R)(\lambda e-RT^0)=0$$

$(I-R)$ is invertible, so $RT^0=\lambda e$.

Theorem 2 QBD process {*Q(t),J(t)*} is positive recurrence if and only if $\rho<1$

Proof Reference to Theorem 3.1.1 in Neuts(1981)[12], the process {*Q(t),J(t)*} is positive recurrence if and only if the spectral radius of $\boldsymbol{R}$, $SP(R)<1$, and the homogeneous equations $XB[R]=0$ has a positive solution.

Because

$$R=\begin{pmatrix} r & 0 & \dfrac{r\theta}{(1-r)c\mu_b} \\ 0 & \dfrac{\lambda}{\lambda+\alpha} & \rho \\ 0 & 0 & \rho \end{pmatrix}$$

we obtain $SP(R)<1$ if and only if $\rho<1$.

In the equations $XB[R]=0$

$$B[R]=\begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C & & \\ & \ddots & \ddots & \ddots & \\ & & B_{c-1} & A_{c-1} & C \\ & & & B & RB+A \end{pmatrix} \tag{6}$$

is the $2+3\times c$-order square, *X* is $2+3\times c$-dimensional row vector. When $\rho<1$, reference to Lemma 2, $RT^0=\lambda e$ and $Be=T^0$, we know that $B[R]$ is a non-periodic irreducible finitely generated, so $XB[R]=0$ has a positive solution.

IV. STEADY-STATE DISTRIBUTION

When $\rho<1$, we assume that (*Q, J*) is the steady-state limit of the QBD process {*Q(t),J(t)*}, let

$$\pi_k=\begin{cases} \pi_{00},\pi_{01}, & k=0 \\ \pi_{k0},\pi_{k1},\pi_{k2}, & k\ge 1 \end{cases} \quad X=\pi_0,\pi_1,\pi_{2,}\cdots,\pi_c,$$

$$\pi_{kj}=P\ Q=k,J=j\ =\lim_{t\to\infty}P\ Q(t)=k,J(t)=j\ ,(k,j)\in\Omega.$$

In order to express the steady-state distribution, we need the following

Lemma 3 There is a relationship between the system parameters and *r*

$$\lambda+\theta+d\mu_v(1-r)=\frac{\theta}{1-r}+d\mu_v=\frac{\lambda}{r} \tag{7}$$

Proof The relationship $RT^0=\lambda e$ in Lemma 2 contains

$$\frac{\theta}{1-r}+d\mu_v=\frac{\lambda}{r} \qquad \theta+d\mu_v(1-r)=\frac{1-r}{r}\lambda$$

on both ends plus λ, we get (7).

Let *a* is an arbitrary constant, recursively define a set of series Ψ_j, $j=0,1,\cdots,d$

$$\Psi_0=1,$$

$$\Psi_j=\frac{\lambda}{j\mu_v}\Psi_{j-1}+\frac{\theta}{j\mu_v}\sum_{i=0}^{j-1}\Psi_i-\frac{\mu_b}{j\mu_v}a,1\le j\le d \tag{8}$$

The series $\{\Psi_j,0\le j\le d\}$ play an important role in the steady-state distribution expression.

The key to obtain the steady-state distribution by the matrix geometric solution method is solving the homogeneous linear equations $XB[R]=0$.

Substituting (4) into $\begin{pmatrix} 0 \\ C \\ RB+A \end{pmatrix}$ of (6):

where

$$RB+A=\begin{pmatrix} -\lambda+\theta+(1-r)d\mu_v & 0 & \dfrac{\theta}{1-r} \\ 0 & -\lambda+\alpha & \lambda+\alpha \\ 0 & 0 & -c\mu_b \end{pmatrix}$$

So the equations $XB[R]=0$ can be written as:

$$\begin{cases} -(\lambda+\theta)\pi_{00}+\mu_v\pi_{10}+\mu_b\pi_{12}=0 \\ \theta\pi_{00}-\lambda\pi_{01}=0 \\ \theta\pi_{10}+\alpha\pi_{11}-(\lambda+\mu_b)\pi_{12}+2\mu_b\pi_{22}=0 \\ \lambda\pi_{k-1,0}-(\lambda+\theta+k\mu_v)\pi_{k0}+\mu_v(k+1)\pi_{k+1,0}=0,1\le k\le d-1 \\ \lambda\pi_{k-1,0}-(\lambda+\theta+d\mu_v)\pi_{k0}+d\mu_v\pi_{k+1,0}=0,d\le k\le c-1 \\ \lambda\pi_{k-1,1}-(\lambda+\alpha)\pi_{k1}=0,1\le k\le c \\ \lambda\pi_{k-1,2}+\theta\pi_{k0}+\alpha\pi_{k1}-(\lambda+k\mu_b)\pi_{k2}+(k+1)\mu_b\pi_{k+1,2}=0,2\le k\le c-1 \\ \lambda\pi_{c-1,0}-[\lambda+\theta+(1-r)d\mu_v]\pi_{c0}=0 \\ \lambda\pi_{c-1,2}+\dfrac{\theta}{1-r}\pi_{c0}+(\lambda+\alpha)\pi_{c1}-c\mu_b\pi_{c2}=0 \end{cases} \tag{9}$$

we mark (9.1~9.9) to each equation in (9).

Lemma 4 $\pi_{k0}=K\Psi_k$, $0\le k\le d-1$, Ψ_k satisfy (8), where $a=\dfrac{\pi_{12}}{\pi_{00}}$, $\boldsymbol{K}$ is an arbitrary constant.

Proof Let $\pi_{00}=K=K\Psi_0$, substituting it into (9.1), we get

$$\pi_{10}=\frac{\lambda+\theta}{\mu_v}\pi_{00}-\frac{\mu_b}{\mu_v}\pi_{12}=\pi_{00}\left(\frac{\lambda}{\mu_v}+\frac{\theta}{\mu_v}-\frac{\mu_b}{\mu_v}a\right)=K\Psi_1$$

when $2\le k\le d-1$, substituting $\pi_{k0}=K\Psi_k$ into (9.4), reference to (8), we have

$$\lambda\pi_{k-1,0} - \lambda+\theta+k\mu_\nu \ \pi_{k0} + \mu_\nu \ k+1 \ \pi_{k+1,0}$$
$$= \lambda\pi_{k-1,0} - \lambda+\theta \ \pi_{k0} - k\mu_\nu \left(\frac{\lambda}{k\mu_\nu}\pi_{k-1,0} + \frac{\theta}{k\mu_\nu}\sum_{i=0}^{k-1}\pi_{i0} - \frac{\mu_b}{k\mu_v}a \right)$$
$$+ \ k+1 \ \mu_\nu \left(\frac{\lambda}{(k+1)\mu_\nu}\pi_{k0} + \frac{\theta}{(k+1)\mu_\nu}\sum_{i=0}^{k}\pi_{i0} - \frac{\mu_b}{(k+1)\mu_v}a \right)$$
$$= -\theta\pi_{k0} - \theta\sum_{i=0}^{k-1}\pi_{i0} + \theta\sum_{i=0}^{k}\pi_{i0} = 0$$

So the Ψ_k is what we need.

Theorem 3 When $\rho < 1$, the steady-state distribution of (Q, J) is

$$\pi_{k0} = \begin{cases} K\Psi_k, & 0 \le k \le d-1 \\ Kr^{k-d}\Psi_d, & k \ge d \end{cases}$$

$$\pi_{k1} = K\frac{\theta}{\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^k, \qquad k \ge 0.$$

$$\pi_{k2} = \begin{cases}
K\dfrac{\theta\Psi_d}{\mu_b(1-r)}\displaystyle\sum_{i=1}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i} + \\
K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=1}^{k}\sum_{v=i}^{d-1}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\Psi_v \\
+K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=2}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}, & 1 \le k \le d. \\
K\dfrac{\theta\Psi_d}{\mu_b(1-r)}\left\{\displaystyle\sum_{i=1}^{d}\begin{array}{l}\dfrac{(i-1)!}{(d+1)!}\left(\dfrac{\lambda}{\mu_b}\right)^{d+1-i} + \\ \displaystyle\sum_{i=d+1}^{k} r^{i-d}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\end{array}\right\} \\
+K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=1}^{d}\sum_{v=i}^{d-1}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d-i+1}\Psi_v \\
+K\dfrac{\theta}{\mu_b}\left\{\begin{array}{l}\displaystyle\sum_{i=2}^{d}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d+1-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1} + \\ \displaystyle\sum_{i=d+1}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\end{array}\right\}, & d+1 \le k \le c. \\
Kr^{c-d}\Psi_d\dfrac{r\theta}{(1-r)c\mu_b}\displaystyle\sum_{i=0}^{k-c-1}r^i\rho^{k-c-1-i} + \\
K\dfrac{\theta}{\lambda}\left(\dfrac{\lambda}{\lambda+\alpha}\right)^c\displaystyle\sum_{i=0}^{k-c-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^i\rho^{k-c-i} \\
+K\dfrac{\theta\Psi_d}{\mu_b(1-r)}\left\{\displaystyle\sum_{i=1}^{d}\begin{array}{l}\dfrac{(i-1)!}{(d+1)!}\left(\dfrac{\lambda}{\mu_b}\right)^{d+1-i} + \\ \displaystyle\sum_{i=d+1}^{c} r^{i-d}\frac{(i-1)!}{c!}\left(\frac{\lambda}{\mu_b}\right)^{c-i}\end{array}\right\}\rho^{k-c} + \\
K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=1}^{d}\sum_{v=i}^{d-1}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d-i+1}\Psi_v\rho^{k-c} + \\
K\dfrac{\theta}{\mu_b}\left\{\begin{array}{l}\displaystyle\sum_{i=2}^{d}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d+1-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1} \\ +\displaystyle\sum_{i=d+1}^{c}\frac{(i-1)!}{c!}\left(\frac{\lambda}{\mu_b}\right)^{c-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\end{array}\right\}\rho^{k-c}, & k > c.
\end{cases}$$

K can be obtained by regularization conditions.

Proof Firstly, derived the solution of $XB[R]=0$.

Reference to Lemma 4,

$$\pi_{k0} = K\Psi_k, 0 \le k \le d-1 \qquad (10)$$

satisfy (9.1) and (9.4). On the other hand, from (9.8) and Lemma 3, we have $\pi_{c0} = r\pi_{c-1,0}$. Let $k = c$-1,from (7), we get $\pi_{c-1,0} = r\pi_{c-2,0}$. Repeated use the equation (9.5), we obtained

$$\pi_{k0} = Kr^{k-d}\Psi_d, d \le k \le c \qquad (11)$$

From (9.2), we have $\pi_{01} = \pi_{00}\dfrac{\theta}{\lambda} = K\dfrac{\theta}{\lambda}$, substituting into (9.6), we can recursive obtain

$$\pi_{k1} = \pi_{k-1,1}\frac{\lambda}{\lambda+\alpha} = \pi_{01}\left(\frac{\lambda}{\lambda+\alpha}\right)^k = K\frac{\theta}{\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^k \ 0 \le k \le c \quad (12)$$

From (9.3), (9.7) and (9.9), we obtained the equations

$$\begin{cases}
\theta\pi_{10} + \alpha\pi_{11} - (\lambda+\mu_b)\pi_{12} + 2\mu_b\pi_{22} = 0 \\
\lambda\pi_{12} + \theta\pi_{20} + \alpha\pi_{21} - (\lambda+2\mu_b)\pi_{22} + 3\mu_b\pi_{32} = 0 \\
\ldots\ldots \\
\lambda\pi_{k-2,2} + \theta\pi_{k-1,0} + \alpha\pi_{k-1,1} - \left[\lambda+(k-1)\mu_b\right]\pi_{k-1,2} + k\mu_b\pi_{k2} = 0 \\
\lambda\pi_{k-1,2} + \theta\pi_{k0} + \alpha\pi_{k1} - (\lambda+k\mu_b)\pi_{k2} + (k+1)\mu_b\pi_{k+1,2} = 0 \\
\ldots\ldots \\
\lambda\pi_{c-2,2} + \theta\pi_{c-1,0} + \alpha\pi_{c-1,1} - \left[\lambda+(c-1)\mu_b\right]\pi_{c-1,2} + c\mu_b\pi_{c2} = 0 \\
\lambda\pi_{c-1,2} + \dfrac{\theta}{1-r}\pi_{c0} + (\lambda+\alpha)\pi_{c1} - c\mu_b\pi_{c2} = 0
\end{cases} \quad (13)$$

Plus each equation in (13), we have

$$\theta\sum_{i=1}^{c}\pi_{i0} + \alpha\sum_{i=1}^{c}\pi_{i1} + \frac{\theta r}{1-r}\pi_{c0} + \lambda\pi_{c1} - \mu_b\pi_{12} = 0 \qquad (14)$$

Substituting (10), (11) and (12) into (14)

$$\pi_{12} = K\left[\frac{\theta}{\mu_b} + \frac{\theta}{\mu_b(1-r)}\Psi_d + \frac{\theta}{\mu_b}\sum_{i=1}^{d-1}\Psi_i\right] \qquad (15)$$

Then superimposed the first k terms of (9.3) and (9.7),we obtained

$$\pi_{k2} = \frac{\lambda}{k\mu_b}\pi_{k-1,2} + \frac{1}{k}\pi_{12} - \frac{\theta}{k\mu_b}\sum_{i=1}^{k-1}\pi_{i0} - \frac{\alpha}{k\mu_b}\sum_{i=1}^{k-1}\pi_{i1} \quad (16)$$

Substituting (10), (11) and (15) into (16), we have

$$\pi_{k2} = \begin{cases}
\dfrac{\lambda}{k\mu_b}\pi_{k-1,2} + K\dfrac{\theta}{k\mu_b}\displaystyle\sum_{i=k}^{d-1}\Psi_i + K\frac{\theta}{k\mu_b(1-r)}\Psi_d + K\frac{\theta}{k\mu_b}\left(\frac{\lambda}{\lambda+\alpha}\right)^{k-1}, & 2 \le k \le d \\
\dfrac{\lambda}{k\mu_b}\pi_{k-1,2} + K\dfrac{\theta}{k\mu_b(1-r)}r^{k-d}\Psi_d + K\dfrac{\theta}{k\mu_b}\left(\dfrac{\lambda}{\lambda+\alpha}\right)^{k-1}, & d+1 \le k \le c
\end{cases}$$

From this and using (15), we recursive obtained

$$\pi_{k2}=\begin{cases} K\dfrac{\theta\Psi_d}{\mu_b(1-r)}\displaystyle\sum_{i=1}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i} \\ +K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=1}^{k}\sum_{v=i}^{d-1}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\Psi_v \\ +K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=2}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}, 1\le k\le d. \\ K\dfrac{\theta\Psi_d}{\mu_b(1-r)}\left\{\displaystyle\sum_{i=1}^{d}\begin{matrix}\dfrac{(i-1)!}{(d+1)!}\left(\dfrac{\lambda}{\mu_b}\right)^{d+1-i}+\\ \displaystyle\sum_{i=d+1}^{k}r^{i-d}\dfrac{(i-1)!}{k!}\left(\dfrac{\lambda}{\mu_b}\right)^{k-i}\end{matrix}\right\}+ \\ K\dfrac{\theta}{\mu_b}\displaystyle\sum_{i=1}^{d}\sum_{v=i}^{d-1}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d-i+1}\Psi_v+ \\ K\dfrac{\theta}{\mu_b}\left\{\begin{matrix}\displaystyle\sum_{i=2}^{d}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d+1-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}+\\ \displaystyle\sum_{i=d+1}^{k}\frac{(i-1)!}{k!}\left(\frac{\lambda}{\mu_b}\right)^{k-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\end{matrix}\right\}, d+1\le k\le c. \end{cases} \quad (17)$$

When $k > c$, from the matrix-geometric solution[13][14], we have

$$\pi_k=(\pi_{k0},\pi_{k1},\pi_{k2})=(\pi_{c0},\pi_{c1},\pi_{c2})R^{k-c}, k\ge c \quad (18)$$

Because

$$R^k=\begin{pmatrix} r^k & 0 & \dfrac{r\theta}{(1-r)c\mu_b}\displaystyle\sum_{i=0}^{k-1}r^i\rho^{k-1-i} \\ 0 & \left(\dfrac{\lambda}{\lambda+\alpha}\right)^k & \displaystyle\sum_{i=0}^{k-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^i\rho^{k-i} \\ 0 & 0 & \rho^k \end{pmatrix}$$

substituting into (18), we obtain

$$\pi_{k0}=Kr^{k-d}\Psi_d, k\ge c$$

$$\pi_{k1}=K\frac{\theta}{\lambda}\left(\frac{\lambda}{\lambda+\alpha}\right)^k, k\ge c$$

$$\pi_{k2}=\pi_{c0}\frac{r\theta}{(1-r)c\mu_b}\sum_{i=0}^{k-c-1}r^i\rho^{k-c-1-i}+\pi_{c1}\sum_{i=0}^{k-c-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^i\rho^{k-c-i}+\pi_{c2}\rho^{k-c}, k\ge c$$

From (17) we have

$$\pi_{c2}=K\frac{\theta\Psi_d}{\mu_b(1-r)}\left\{\sum_{i=1}^{d}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d+1-i}+\sum_{i=d+1}^{c}r^{i-d}\frac{(i-1)!}{c!}\left(\frac{\lambda}{\mu_b}\right)^{c-i}\right\}+$$

$$K\frac{\theta}{\mu_b}\sum_{i=1}^{d}\sum_{v=i}^{d-1}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d-i+1}\Psi_v+$$

$$K\frac{\theta}{\mu_b}\left\{\sum_{i=2}^{d}\frac{(i-1)!}{(d+1)!}\left(\frac{\lambda}{\mu_b}\right)^{d+1-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}+\sum_{i=d+1}^{c}\frac{(i-1)!}{c!}\left(\frac{\lambda}{\mu_b}\right)^{c-i}\left(\frac{\lambda}{\lambda+\alpha}\right)^{i-1}\right\}$$

The ***K*** in the theorem can be obtained by regularization conditions, so the theorem is proved.

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