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Fuzzy Closed Ideals of B-Algebras

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***Abstract*— In this paper, we apply the concept of fuzzy set to ideals and closed ideals in B-algebras. The notion of a fuzzy closed ideal of a B-algebra is introduced, and some related properties are investigated. Also, the product of fuzzy B-algebra is investigated.**
***Keywords*—B-algebras, fuzzy sets, fuzzy ideals, homomorphism, fuzzy closed ideals, homomorphism, equivalence relation, level cut, product of B-algebra.**

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I. INTRODUCTION

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [5]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. Hu and Li [4] and Iseki [7] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebra is a proper subclass of the BCH-algebras. Neggers and Kim [10, 11] introduced a new notion, called a B-algebra which is related to several classes of algebras of interest such as BCH/BCI/BCK-algebras. Cho and Kim [2] discussed further relations between B-algebras and other topics especially quasigroups. Park and Kim [12] obtained that every quadratic B-algebra on a field X with $|X| \geq 3$ is a BCI-algebra. Jun et al. [8] fuzzyfied (normal) B-algebras and gave a characterization of a fuzzy B-algebras.
Saeid introduced fuzzy topological B-algebras. For the general development of the B-algebras, the ideal theory plays an important role.

In this paper, some extended result of fuzzy ideal called FC-ideal of B-subalgebras is presented based on fuzzy sets, and obtain some results on them. At the same time, the notion of equivalence relations on the family of all fuzzy ideals of a B-algebra are presented and investigated some related properties. The product of fuzzy B-algebra has been introduced and some important properties are of it are also studied.
The rest of this paper is organized as follows. The following section briefly reviews some background on B-algebra, B-subalgebra, fuzzy set, fuzzy B-subalgebras. In Section III, we propose the concepts and operations of fuzzy ideal and fuzzy closed ideal and discuss their properties in detail. In Section IV, we investigate properties of fuzzy ideals under homomorphisms. In Section V, we introduce equivalence relations on fuzzy ideals. In Section VI, product of fuzzy B-algebra and some of its properties are studied. Finally, in Section VII, we draw the conclusion and present some topics for future research.

II. PRELIMINARIES

In this section, some elementary aspects that are necessary for this paper are included.

An algebra $(X; *, 0)$ of type (2, 0) is called a B-algebra [10] if it satisfies the following axioms:

B1. $x * x = 0$
B2. $x * 0 = x$
B3. $(x * y) * z = x * (z * (0 * y))$, for all $x, y \in X$.

Example 2.1 [10] Let X be the set of all real numbers except for a negative integer -n. Define a binary operation $*$ on X by

$$x * y = \frac{n(x-y)}{n+y} .$$

Then $(X, *, 0)$ is a B-algebra.
Lemma 2.2 [2] If X is a B-algebra, then $0 * (0 * x) = x$ for all $x \in X$.
Lemma 2.3 [10] If X is a B-algebra, then $(x * y) * (0 * y) = x$ for all $x, y \in X$.

A non-empty subset S of a B-algebra X is called a subalgebra ([11]) of X if $x * y \in S$ for any $x, y \in S$. A mapping $f: X \to Y$ of B-algebra is called a homomorphism ([11]) if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f: X \to Y$ is a B-homomorphism, then $f(0) = 0$. A partial ordering "$\leq$ '' on X can be defined by $x \leq y$ if and only if $x * y = 0$.

Let X be the collection of objects denoted generally by x then a fuzzy set [14] A in X is defined as $A = \{< x, \alpha_A(x) >: x \in X\}$ where $\alpha_A(x)$ is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$. The complement of A is denoted by A^c and is given by $A^c = \{< x, \alpha_A{}^c(x) >: x \in X\}$ where $\alpha_A{}^c(x) = 1 - \mu_A(x)$. Combined the definition of B-subalgebra over crisp set and the idea of fuzzy set Ahn and Bang [1] defined fuzzy B-subalgebra, which is defined below.

Definition 2.4 [1] A fuzzy set A in X is called a fuzzy B-algebra if it satisfies the inequality $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ for all $x, y \in X$.

III. FUZZY CLOSED IDEALS OF B-ALGEBRAS

In this section, fuzzy ideal and fuzzy closed ideal of B-algebra are defined and some propositions and theorems are presented. In what follows, let X denote a B-algebra unless otherwise specified.

Definition 3.1 Let A be a fuzzy set in a B-algebra X. Then A is called a fuzzy ideal of X if for all $x, y \in X$ it satisfies:

(FI1) $\alpha_A(0) \geq \alpha_A(x)$

(FI2) $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$.

Example 3.2 Let X= {0, 1, 2, 3} be a set with the following Cayley table:

TABLE I

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define a fuzzy set A in X by $\alpha_A(0) = \alpha_A(2) = 1$and $\alpha_A(1) = \alpha_A(3) = m$, where $m \in [0, 1)$. Then A is a fuzzy-ideal of X.

Definition 3.3 A fuzzy set A in X is called a fuzzy closed ideal of X if it satisfies (FI1), (FI2) along with (FI3) $\alpha_A(0 * x) \geq \alpha_A(x)$ for all $x \in X$.

Example 3.4 Let X= {0, 1, 2, 3, 4, 5} be a set with the following Cayley table:

TABLE II

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Then $(X, *, 0)$ is a B-algebra (see [11], Example 3.5). We define A in X by, $\alpha_A(0) = 0.6$, $\alpha_A(1) = \alpha_A(2) = 0.5$ and $\alpha_A(3) = \alpha_A(4) = \alpha_A(5) = 0.4$. By routine calculations, one can verify that A is fuzzy closed ideal of X.

Proposition 3.5 Every fuzzy closed ideal is a fuzzy ideal.

The converse of above Proposition is not true in general as seen in the following example.

Example 3.6 Consider a B-algebra X = {0, 1, 2, 3, 4, 5} with the table

TABLE III

*	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

Let us define an fuzzy set $A = (\alpha_A, \beta_A)$ in X by $\alpha_A(0) = 0.6$, $\alpha_A(1) = 0.5$ and $\alpha_A(2) = \alpha_A(3) = \alpha_A(4) = \alpha_A(5) = 0.4$. We know that A is a fuzzy ideal of X. But it is not an fuzzy closed ideal of X since $\alpha_A(0 * x) < \alpha_A(x)$ for some $x \in X$.

Corollary 3.7 Every fuzzy B-subalgebra satisfying (FI2) is a fuzzy closed ideal.

Theorem 3.8 Every fuzzy closed ideal of a B-algebra X is a fuzzy B-subalgebra of X.

Proof: If A is a fuzzy closed ideal of X, then for any $x \in X$ we have $\alpha_A(0 * x) \geq \alpha_A(x)$. Now

$\alpha_A(x * y) \geq \min\{\alpha_A((x * y) * (0 * y)), \alpha_A(0 * y)\}$, by (FI2)

$= \min\{\alpha_A(x), \alpha_A(0 * y)\}$

$\geq \min\{\alpha_A(x), \alpha_A(y)\}$, by (FI3)

Hence the theorem.

Proposition 3.9 If a fuzzy set A in X is a fuzzy closed ideal, then for all $x \in X$, $\alpha_A(0) \geq \alpha_A(x)$.

Proof: Straightforward.

Definition 3.10 [15] Let $A = \{< x, \alpha_A(x) >: x \in X\}$ and $B = \{< x, \alpha_B(x) >: x \in X\}$ be two fuzzy sets on X. Then the intersection of A and B is denoted by $A \cap B$ and is given by $A \cap B = \{< x, \min\{\alpha_A(x), \alpha_B(x)\} >: x \in X\}$.

The intersection of two fuzzy ideals of a B-algebra is also a fuzzy ideal of a B-algebra, which is proved in the following theorem.

Theorem 3.11 Let A_1 and A_2 be two fuzzy ideals of a B-algebras X. Then $A_1 \cap A_2$ is also fuzzy ideal of B-algebra X.

Proof: Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now, $\alpha_{A1 \cap A2}(0) = \alpha_{A1 \cap A2}(x * x) \geq \min\{\alpha_{A1 \cap A2}(x), \alpha_{A1 \cap A2}(x)\} = \alpha_{A1 \cap A2}(x)$. Also, $\alpha_{A1 \cap A2}(x) = \min\{\alpha_{A1}(x), \alpha_{A2}(x)\}$,

$\geq \min\{\min\{\alpha_{A1}(x * y), \alpha_{A1}(y)\}, \min\{\alpha_{A2}(x * y), \alpha_{A2}(y)\}\}$

$= \min\{\min\{\alpha_{A1}(x * y), \alpha_{A2}(x * y)\}, \min\{\alpha_{A1}(y), \alpha_{A2}(y)\}\}$

$= \min\{\alpha_{A1 \cap A2}(x * y), \alpha_{A1 \cap A2}(y)\}$.

Hence, $A_1 \cap A_2$ is a fuzzy ideal of a B-algebra of X.

The above theorem can be generalized as follows.

Theorem 3.12 Let $\{A_i \mid i = 1, 2, 3, 4 \ldots\}$ be a family of fuzzy ideals of a B-algebra X. Then $\cap A_i$ is also a fuzzy ideal of B-algebra X where, $\cap A_i = \{< x, \min \alpha_{Ai}\{(x) >: x \in X\}$.

Theorem 3.13 Let A be a fuzzy ideals of a B-algebras X. If $x*y \leq z$ then $\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}$.

Proof: Let $x, y, z \in X$ such that $x * y \leq z$ .Then $(x * y) * z = 0$, and thus $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\}$

$\geq \min\{\min\{\alpha_A((x*y)*z), \alpha_A(z)\}, \alpha_A(y)\}$

$= \min\{\min\{\alpha_A(0), \alpha_A(z)\}, \alpha_A(y)\}$

$= \min\{\alpha_A(y), \alpha_A(z)\}$.

Theorem 3.14 Let A be a fuzzy ideals of a B-algebras X. If $x \leq y$ then $\alpha_A(x) \geq \alpha_A(y)$ i.e, order reversing.

Proof: Let x, y ∈ X such that $x \leq y$ .Then $x * y = 0$, and thus $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\} = \min\{\alpha_A(0), \alpha_A(y)\} = \alpha_A(y)$.

The above lemma can be generalized as

Lemma 3.15 Let A be a fuzzy ideal of X, then $(....((x*a_1)*a_2)*....)*a_n = 0$ for any $x, a_1, a_2, ..., a_n \in X$, implies $\alpha_A(x) \geq \min\{\alpha_A(a_1), \alpha_A(a_2), ..., \alpha_A(a_n)\}$.

Proof: Using induction on n and by Lemma 3.13 and Lemma 3.14 we can easily prove the theorem.

Theorem 3.16 Let B be a crisp subset of X. Suppose that $A = \{< x, \alpha_A(x) >: x \in X\}$ is a fuzzy set in X defined by $\alpha_A(x) = \lambda$ if $x \in B$ and $\alpha_A(x) = \tau$ if $x \notin B$ for all $\lambda, \tau \in [0, 1]$ with $\lambda \geq \tau$. Then A is a fuzzy closed ideal of X if and only if B is a closed ideal of X.

Proof: Assume that A is a fuzzy closed ideal of X. Let x ∈ B. Then, by (FI3), we have $\alpha_A(0*x) \geq \alpha_A(x) = \lambda$ and so $\alpha_A(0*x) = \lambda$. It follows that $0*x \in B$. Let x,y ∈ X be such that $x*y \in B$ and y ∈ B. Then $\alpha_A(x*y) = \lambda = \alpha_A(y)$, and hence $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\} = \lambda$. Thus $\alpha_A(x) = \lambda$, that is, x ∈ B. Therefore B is a closed ideal of X.

Conversely, suppose that B is a closed ideal of X. Let x ∈ X. If x ∈ B, then $0*x \in B$ and thus $\alpha_A(0*x) = \lambda = \alpha_A(x)$. If $x \notin B$, then $\alpha_A(x) = \tau \leq \alpha_A(0*x)$. Let x,y ∈ X. If $x*y \in B$ and y ∈ B, then x ∈ B. Hence, $\alpha_A(x) = \lambda = \min\{\alpha_A(x*y), \alpha_A(y)\}$. If $x*y \notin B$ and $y \notin B$, then clearly $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\}$. If exactly one of $x*y$ and y belong to B, then exactly one of $\alpha_A(x*y)$ and $\alpha_A(y)$ is equal to τ. Therefore, $\alpha_A(x) \geq \tau = \min\{\alpha_A(x*y), \alpha_A(y)\}$. Consequently, A is a fuzzy closed ideal of X.

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

Definition 3.17 Let A is a fuzzy B-subalgebra of X. For s ∈ [0, 1], the set $U(\alpha_A : s) = \{x \in X : \alpha_A(x) \geq s\}$ is called upper s-level of A.

Theorem 3.18 A fuzzy set A is a closed ideal of X if and only if the set $U(\alpha_A : s)$ is closed ideal of X for every s ∈ [0, 1].

Proof: Suppose that A is a fuzzy closed ideal of X. For s ∈ [0, 1], obviously, $0 * x \in U(\alpha_A : s)$, where x ∈ X. Let x, y ∈ X be such that $x * y \in U(\alpha A : s)$ and $y \in U(\alpha_A : s)$. Then $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\} \geq s$. Then $x \in U(\alpha_A : s)$. Hence, $U(\alpha_A : s)$ is closed ideal of X.

Conversely, assume that each non-empty level subset $U(\alpha_A : s)$ is closed ideals of X. For any x ∈ X, let $\alpha_A(x) = s$. Then $x \in U(\alpha_A : s)$. Since $0 * x \in U(\alpha_A : s)$, it follows that $\alpha_A(0*x) \geq s = \alpha_A(x)$, for all x ∈ X.

If there exist λ, κ ∈ X such that $\alpha_A(\lambda) < \min\{\alpha_A(\lambda * \kappa), \alpha_A(\kappa)\}$, then by taking $s' = \frac{1}{2}[\alpha_A(\lambda * \kappa) + \min\{\alpha_A(\lambda), \alpha_A(\kappa)\}]$, it follows that $\lambda * \kappa \in U(\alpha_A : s')$ and $\kappa \in U(\alpha_A : s')$, but $\lambda \notin U(\alpha_A : s')$, which is a contradiction. Hence, $U(\alpha_A : s')$ is not closed ideal of X. Hence, A is a fuzzy closed ideal of X.

Theorem 3.19 Let A be a closed ideal of X with the finite image. Then every descending chain of closed ideals of X terminates at finite step.

Proof: Suppose that there exists a strictly descending chain $D_0 \supset D_1 \supset D_2 \supset \cdots$ of closed ideals of X which does not terminate at finite step. Define a fuzzy set A in X by $\alpha_A(x) = \frac{n}{n+1}$ if $x \in D_n \backslash D_{n+1}$, $n = 0, 1, 2, ….$ and $\alpha_A(x)=1$ if $x \in \bigcap_{n=0}^{\infty} D_n$, where $D_0 = X$. We prove that A is a fuzzy closed ideal of X. It is easy to show that A satisfies (FI3). Let x, y ∈ X. Assume that $x * y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots$,Without loss of generality, we may assume that $n \leq k$. Then obviously $x * y$ and $y \in D_n$, so $x \in D_n$ because D_n is a closed ideal of X. Hence

$$\alpha_A(x) \geq \frac{n}{n+1} = \min\{\alpha_A(x * y), \alpha_A(y)\}.$$

If $x * y$, $y \in \bigcap_{n=0}^{\infty} D_n$, then $x \in \bigcap_{n=0}^{\infty} D_n$. Thus

$$\alpha_A(x) \geq 1 = \min\{\alpha_A(x * y), \alpha_A(y)\}.$$

If $x * y \notin \bigcap_{n=0}^{\infty} D_n$ and $y \in \bigcap_{n=0}^{\infty} D_n$, then there exists k ∈ N (the set of natural numbers) such that $x * y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$\alpha_A(x) \geq \frac{k}{k+1} = \min\{\alpha_A(x * y), \alpha_A(y)\}.$$

Finally suppose that $x * y \in \bigcap_{n=0}^{\infty} D_n$ and $y \notin \bigcap_{n=0}^{\infty} D_n$.Then $y \in D_r \setminus D_{r+1}$ for some r ∈ N. Hence $x \in D_r$, and so

$$\alpha_A(x) \geq \frac{r}{r+1} = \min\{\alpha_A(x * y), \alpha_A(y)\}.$$

Consequently, we conclude that A is a closed ideal of X and A has infinite number of different values, which is a contradiction. This completes the proof.

Theorem 3.20 The following are equivalent:

(i) every ascending chain of closed ideals of X terminates at finite step.

(ii) the set of values of any fuzzy closed ideals is a well-ordered subset of [0, 1].

Proof: Let A be a fuzzy closed ideals of X. Suppose that the set of values of A is not a well-ordered subset of [0, 1]. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\mu_A(x) = \alpha_n$. Let $D_n = \{x \in X | \mu_A(x) \geq \alpha_n\}$. Then $D_1 \subset D_2 \subset D_3 \subset \cdots$ is a strictly ascending chain of closed ideals of X which is not terminating. This is a contradiction.

Conversely, suppose that there exists a strictly ascending chain

$$D_1 \subset D_2 \subset D_3 \subset \cdots \text{———} (*)$$

of closed ideals of X which does not terminate at finite step. Define a fuzzy set A in X by $\alpha_A(x) = \frac{1}{k}$ where $k = \min\{n \in N | x \in D_n\}$ and $\alpha_A(x) = 0$ if $x \notin D_n$, where $D = \bigcup_{n \in N} D_n$.

We prove that A is a closed ideal of X. It is easy to show that A satisfies (FI3). Let x, y ∈ X. If $x * y, y \in D_n \backslash D_{n-1}$ for n = 2, 3, · · · , then $x \in D_n$. It follows that

$$\alpha_A(x) \geq \frac{1}{n} = \min\{\alpha_A(x * y), \alpha_A(y)\}.$$

Assume that $x * y \in D_n$ and $y \in D_n \backslash D_m$ for all m < n. Since A is a closed ideals of X, therefore, x ∈ Dn. Thus

$$\alpha_A(x) \geq \frac{1}{n} \geq \frac{1}{m+1} \geq \alpha_A(y)$$

and hence, $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$.

Similarly, for the case $x * y \in D_n \backslash D_m$ and $y \in D_n$, we have, $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$.

Hence A is a closed ideal of X. Since the chain (*) is not terminating, A has strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well-ordered.

IV. Investigation of Fuzzy Ideals under Homomorphisms

In this section, homomorphism of fuzzy B-algebra is defined and some results are studied.

Let f be a mapping from the set X into the set Y. Let B be a fuzzy set in Y. Then the inverse image of B, is defined as $f^{-1}(B) = \{< x, f^{-1}(\alpha_B)(x) >: x \in X\}$ with the membership is given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$. It can be shown that $f^{-1}(B)$ is a fuzzy set.

Theorem 4.1 Let f: X → Y be a homomorphism of B-algebras. If B is a fuzzy ideal of Y, then the pre-image $f^{-1}(B)$ of B under f in X is a fuzzy ideal of X.

Proof: For all x ∈ X, $f^{-1}(\alpha_B)(x) = \alpha_B(f(x)) \leq \alpha_B(0) = \alpha_B(f(0)) = f^{-1}(\alpha_B)(0)$. Let x, y ∈ X. Then

$$\begin{aligned} f^{-1}(\alpha_B)(x) &= \alpha_B(f(x)) \\ &\geq \min\{\alpha_B((f(x) * f(y)), \alpha_B(f(y))\} \\ &\geq \min\{\alpha_B(f(x * y), \alpha_B(f(y))\} \\ &= \min\{f^{-1}(\alpha_B)(x * y), f^{-1}(\alpha_B)(y)\} \end{aligned}$$

Hence, $f^{-1}(B) = \{< x, f^{-1}(\alpha_B)(x) >: x \in X\}$ is a fuzzy ideal of *X*.

Theorem 4.2 Let f: X → Y be a homomorphism of B-algebras. Then B is a fuzzy ideal of Y, if $f^{-1}(B)$ of B under f in X is a fuzzy ideal of X.

Proof: For any x ∈ Y , ∃ a ∈ X such that f(a) = x. Then$\alpha_B(x) = \alpha_B(f(a)) = f^{-1}(\alpha_B)(a) \leq f^{-1}(\alpha_B)(0) = \alpha_B(f(0)) = \alpha_B(0)$.

Let x, y ∈ Y . Then f(a) = x and f(b) = y for some a, b ∈ X. Thus

$$\begin{aligned} \alpha_B(x) &= \alpha_B(f(a)) = f^{-1}(\alpha_B)(a) \\ &\geq \min\{f^{-1}(\alpha_B)(a * b), f^{-1}(\alpha_B)(b)\} \\ &= \min\{\alpha_B(f(a * b)), \alpha_B(f(b))\} \\ &= \min\{\alpha_B(f(a) * f(b)), \alpha_B(f(b))\} \\ &= \min\{\alpha_B(x * y), \alpha_B(y)\}. \end{aligned}$$

Then B is a fuzzy ideal of Y.

V. Equivalence Relations on Fuzzy Ideals

Let FI(X) denote the family of all fuzzy ideals of X and let ρ ∈ [0, 1]. Define binary relation U^ρ on FI(X) as follows: $(A,B) \in U^\rho \Leftrightarrow U(\alpha_A : \rho) = U(\alpha_B : \rho)$ for A in FI(X). Then clearly U^ρ is equivalence relations on FI(X). For any A ∈ FI(X), let $[A]_{U\rho}$ denote the equivalence class of A modulo U^ρ, and denote by FI(X)/ U^ρ, the collection of all equivalence classes modulo U^ρ, i.e., FI(X)/ $U^\rho := \{[A]_{U\rho} | A \in FI(X)\}$. These set is also called the quotient set.

Now let T(X) denote the family of all ideals of X and let $\rho = [\rho_1, \rho_2] \in [0, 1]$. Define mappings fρ from FI(X) to T(X) ∈{ϕ} by $f_\rho(A) = U(\alpha_A : \rho)$ for all A ∈ FI(X). Then f_ρ is clearly well-defined.

Theorem 5.1 For any ρ ∈ [0, 1], the map f_ρ is surjective from FI(X) to T(X) ∈ {ϕ}.

Proof: Let ρ ∈ [0, 1]. Obviously $f_\rho(0) = U(0 : \rho) = \phi$. Let P (≠ ϕ) ∈ FI(X). For $P_\sim = \{< x, \chi_P(x) >: x \in X\} \in FI(X)$, we have $f_\rho(P\sim) = U(\chi_P : \rho) = P$. Hence f_ρ is surjective.

Theorem 5.2 the quotient set FI(X)/U^ρ is equipotent to T(X) ∈ {ϕ} for every ρ ∈ [0, 1].

Proof: For ρ ∈ [0, 1] let f_ρ*be a map from FI(X)/U^ρ to T(X) ∪ {ϕ} defined by $f_\rho^*([A]_{U\rho}) = f_\rho(A)$ for all A ∈ FI(X)}. If $U(\alpha_A : \rho) = U(\alpha_B : \rho)$ for A and B in FI(X), then (A,B) ∈ U^ρ; hence $[A]_{U\rho}=[B]_{U\rho}$. Therefore the maps f_ρ*is injective. Now let P (≠ ϕ) ∈ FI(X). For $P_\sim$ ∈ FI(X), we have $f_\rho^*([P_\sim]_{U\rho}) = f\rho(P_\sim) = U(\chi_P : \rho) = P$. Finally, for 0 ∈ FI(X) we get $f_\rho^*([0]_{U\rho}) = f_\rho(0) = U(0 : \rho) = \phi$. This shows that f_ρ* is surjective. This completes the proof.

VI. Product of Fuzzy B-Algebra

In this section, product of fuzzy ideals in B-algebra is defined and some results are studied.

Definition 6.1 Let A= {< x, $\alpha_A(x)$ >: x ∈ X} and B = {< x, $\alpha_B(x)$ >: x ∈ X} be two fuzzy sets on X. The Cartesian product A × B = {<(x,y), $\alpha_A \times \alpha_B$ (x,y) >: x,y ∈ X}is defined by $(\alpha_A \times \alpha_B)(x, y) = \min\{\alpha_A(x), \alpha_B(y)\}$ where $\alpha_A \times \alpha_B : X \times X \to [0, 1]$ for all x, y ∈ X.

Theorem 6.2 Let A and B be fuzzy ideals of X, then A × B is a fuzzy ideal of X × X.

Proof: For any (x, y) ∈ X × X, we have

$$\begin{aligned} (\alpha_A \times \alpha_B)(0, 0) &= \min\{\alpha_A(0), \alpha_B(0)\} \\ &\geq \min\{\alpha_A(x), \alpha_B(y)\}, \text{ for all } x, y \in X, \\ &= (\alpha_A \times \alpha_B)(x, y). \end{aligned}$$

Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then

$$\begin{aligned} (\alpha_A \times \alpha_B)(x_1, y_2) &= \min\{\alpha_A(x_1), \alpha_B(y_1)\} \\ &\geq \min\{\min\{\alpha_A(x_1 * x_2), \alpha_A(x_2)\}, \min\{\alpha_B(y_1 * y_2), \alpha_B(y_2)\}\} \\ &= \min\{\min\{\alpha_A(x_1 * x_2), \alpha_B(y_1 * y_2)\}, \min\{\alpha_A(x_2), \alpha_B(y_2)\}\} \\ &= \min\{(\alpha_A \times \alpha_B)(x_1 * x_2, y_1 * y_2), (\alpha_A \times \alpha_B)(x_2, y_2)\} \\ &= \min\{(\alpha_A \times \alpha_B)((x_1, y_1) * (x_2, y_2)), (\alpha_A \times \alpha_B)(x_2, y_2)\}. \end{aligned}$$

Hence, A × B is a fuzzy ideal of X × X.

The converse of Theorem 6.2 may not be true as seen in the following example.

Example 6.3 Let s, t ∈ [0; 1) such that s ≤ t. Define fuzzy sets A and B in X by $\alpha_A(x) = s$, $\alpha_B(x) = t$ if x = 0 and $\alpha_B(x) = 1$ otherwise, for all x ∈ X, respectively.

If x ≠ 0, then $\alpha_B(x) = 1$, and thus

$(\alpha_A \times \alpha_B)(x, x) = \min\{\alpha_A(x), \alpha_B(x)\} = \min\{s, 1\} = s$.

If x = 0, then $\alpha_B(x) = t < 1$, and thus

$(\alpha_A \times \alpha_B)(x, x) = \min\{\alpha_A(x), \alpha_B(x)\} = \min\{s, t\} = s$.

That is, A × B is a constant function and so A × B is a fuzzy ideal of X×X. Now A is a fuzzy ideal of X, but B is not a fuzzy ideal of X since for $x \neq 0$, we have $\alpha_B(0) = t < 1 = \alpha_B(x)$.

Proposition 6.4 Let A and B are fuzzy closed ideals of X, then A × B is a fuzzy closed ideal of X × X.

Proof: Now,

$$\begin{aligned}(\alpha_A \times \alpha_B)((0, 0) * (x, y)) &= (\alpha_A \times \alpha_B)(0 * x, 0 * y)\\ &= \min\{\alpha_A(0 * x), \alpha_B(0 * y)\}\\ &\geq \min\{\alpha_A(x), \alpha_B(y)\}\\ &= (\alpha_A \times \alpha_B)(x, y).\end{aligned}$$

Hence, A × B is a fuzzy closed ideal of X × X.

Definition 6.5 Let A and B is fuzzy ideals of X. For $s \in [0, 1]$, the set $U(\alpha_A\times\alpha_B : s) = \{(x, y) \in X\times X | (\alpha_A\times\alpha_B)(x, y) \geq s\}$ is called upper s-level of A × B.

Lemma 6.6 Let A and B be fuzzy sets in X such that A × B is a fuzzy ideal of X × X, then

(i) Either $\alpha_A(0) \geq \alpha_A(x)$ or $\alpha_B(0) \geq \alpha_B(x)$ for all $x \in X$.

(ii) If $\alpha_A(0) \geq \alpha_A(x)$ for all $x \in X$, then either $\alpha_B(0) \geq \alpha_A(x)$ or $\alpha_B(0) \geq \alpha_B(x)$.

(iii)) If $\alpha_B(0) \geq \alpha_B(x)$ for all $x \in X$, then either $\alpha_A(0) \geq \alpha_A(x)$ or $\alpha_A(0) \geq \alpha_B(x)$.

Proof: (i) Assume that $\alpha_A(x) > \alpha_A(x)$ and $\alpha_B(y) > \alpha_B(0)$ for some $x, y \in X$. Then $(\alpha_A \times \alpha_B)(x, y) = \min \{\alpha_A(x), \alpha_B(y)\} > \min \{\alpha_A(0), \alpha_B(0)\} = (\alpha_A \times \alpha_B)(0, 0)$ which implies $(\alpha_A \times \alpha_B)(x, y) > (\alpha_A \times \alpha_B)(0, 0)$ for all $x, y \in X$, which is a contradiction. Hence (i) is proved.

(ii) Again assume that $\alpha_B(0) < \alpha_A(x)$ and $\alpha_B(0) < \alpha_B(y)$ for all $x, y \in X$. Then $(\alpha_A \times \alpha_B)(0, 0) = \min \{\alpha_A(0), \alpha_B(0)\} = \alpha_B(0)$. Now, $(\alpha_A \times \alpha_B)(x, y) = \min \{\alpha_A(x), \alpha_B(y)\} > \alpha_B(0) = (\alpha_A \times \alpha_B)(0, 0)$, which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

Theorem 6.7 For any fuzzy set A and B, A × B is a fuzzy ideal of X × X if and only if the non-empty upper s-level cut $U(\alpha_A \times \alpha_B : s)$ is closed ideal of X × X for any $s \in [0, 1]$.

Proof: Let A and B are fuzzy closed ideals of X, therefore for any $(x, y) \in X\times X$, $(\alpha_A \times \alpha_B)((0, 0) * (x, y)) \geq (\alpha_A \times \alpha_B)(x, y)$.

For $s \in [0, 1]$, if $(\alpha_A \times \alpha_B)(x, y) \geq s$.

That is, $(\alpha_A \times \alpha_B)((0, 0) * (x, y)) \geq s$

This implies, $(0, 0) * (x, y) \in U(\alpha_A \times \alpha_B : s)$.

Let $(x, y), (x', y') \in X\times X$ such that $(x, y)*(x', y') \in U(\alpha_A\times\alpha_B : s)$ and $(x', y') \in U(\alpha_A\times\alpha_B : s)$. Now, $(\alpha_A \times \alpha_B)(x, y) \geq \min\{(\alpha_A \times \alpha_B)((x, y) * (x', y')), (\alpha_A \times \alpha_B)(x', y')\} \geq \min(s, s) = s$. This implies, $(x, y) \in U(\alpha_A \times \alpha_B : s)$. Thus $U(\alpha_A \times \alpha_B : s)$ is closed ideal of X × X.

Conversely, let $(x, y) \in X \times X$ such that $(\alpha_A \times \alpha_B)(x, y) = s$. This implies, $(x, y) \in U(\alpha_A\times\alpha_B : s)$. Since $(0, 0)*(x, y) \in U(\alpha_A\times\alpha_B : s)$ (by definition of closed ideal). Therefore, $(\alpha_A \times \alpha_B)((0, 0) *(x, y)) \geq s$. This gives, $(\alpha_A\times\alpha_B)((0, 0)*(x, y)) \geq (\alpha_A\times\alpha_B)(x, y)$. Hence, A × B is a fuzzy closed ideal of X × X.

VII. CONCLUSIONS

In the present paper, we have presented some extended results of fuzzy ideal called fuzzy closed ideals of B-algebras and investigated some of their useful properties. The product of B-subalgebra has been introduced and some important properties are of it are also studied. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BF-algebras, lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of B-algebras.

In our future study of fuzzy structure of B-algebra, may be the following topics should be considered:

- To find T-fuzzy closed ideals of B-algebras, where T are given imaginable triangular norm,
- To get more results in fuzzy closed ideals of B-algebra and application,
- To find $(\in, \in \vee q)$-fuzzy ideals of B-algebras

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